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MHV Lagrangian for $N = 4$ super Yang-Mills

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ABSTRACT: Here we formulate two field redefinitions for N=4 Super Yang-Mills in light cone superspace that generates only MHV vertices in the new Lagrangian. After careful consideration of the S-matrix equivalence theorem, we see that only the canonical transformation gives the MHV Lagrangian that would correspond to the CSW expansion. Being in superspace, it is easier to analyse the equivalence theorem at loop level. We calculate the on shell amplitude for 4pt ($\bar{\Lambda}\bar{\Lambda}\Lambda\Lambda$) MHV in the new lagrangian and show that it reproduces the previously known form. We also briefly discuss the relationship with the off-shell continuation prescription of CSW.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry

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Contents

1 Introduction

N=4 Super Yang-Mills theory has been an extensive field of study ever since its introduction in 1977 [1]. The large amount of symmetry has proven to be both a blessing, being a finite theory and making connections to string theory and integrability, and an obstacle. With the failure of numerous attempt to construct its off-shell formulation, in recent years the attention had turned to on-shell methods for the S-matrix of the theory, see [2, 3] for review. Some of the on-shell methods developed has also been utilized in less symmetric theories. One of the ingredients is the work of Cachazo, Svrcek and Witten (CSW) [4]. From the construction of N=4 SYM tree amplitudes in terms of twistor superstring [5], they propose a new perturbative approach to construct YM amplitudes based on using the on-shell form of MHV(Maximal Helicity Violating) amplitudes as vertices. Constructed originally for YM it was also valid for N=4 SYM [6]. Such an approach was also used for loop amplitudes using the cut constructible nature of N=4 SYM [7].

Various efforts has been made on providing a proof for the CSW program. Risager [8] showed that the CSW program is just a result of certain recursion relationship similar to that developed by Britto, Cachazo and Feng [9], which uses the fact that one can use unitarity to relate one loop amplitudes to tree amplitudes, while infrared consistency conditions relate different tree amplitudes to satisfy a recursion relationship. However, in the proof for the BCFW recursion relationship [10] one actually uses the CSW program to prove the behavior of tree amplitudes in certain limits. Recently, one has been able to prove that BCFW eventually leads to the CSW expansion [11].

Even though the relation between various on-shell methods has become clear, one would still like to see it's relationship to the action approach of QFT, since originally the theory was defined by its Lagrangian. Making the connection may well shed light on what properties of the Lagrangian leads to such simple structures for it's scatering amplitudes. Effort along this line of thought began by Gorsky and Rosly [12] where they propose a non-local field redefinition to transform the self-dual part of the YM action into a free action, while the remaining vertices will transform into an infinite series of MHV vertices. In this sense the MHV lagrangian can be viewed as a perturbation around the self-dual sector of ordinary Yang-Mills. This seems natural since self-dual Yang-Mills is essentially a free theory classically. Yang-Mills lagrangian in light-cone (or space-cone [13]) gauge is a natural framework for such a field redefinition since the positive and negative helicity component of the gauge field is connected by a scalar propagator. Work on the light-cone action began by Mansfield [14] emphasizing on the canonical nature of the field redefinition, the formulation was also extended to massless fermions. The explicit redefinition for Yang-Mills was worked out by Ettle and Morris [15]. The canonical condition in [14, 15] ensures that using the field redefinition complications will not arise when taking into account of currents in computing scattering amplitude. This will not be true for more general field redefinitions as we show in this letter.

The progress above was mostly done in the frame work of ordinary Yang-Mills. However, the CSW program has also achieved various success in N=4 SYM as priorly mentioned. It is also interesting in [15] the redefinition for positive and negative helicity have very similar form which begs for a formulation putting them on equal footing. This formulation is present in N=4 light-cone superspace [16] where both the positive and negative helicity gauge field sits on opposite end of the multiplet contained in a single chiral superfield. Thus a field redefinition for one superfield contains the redefinition for the entire multiplet, which would be very difficult if one try the CSW program for the component fields separately. Moreover, N=4 Self-dual YM is free at quantum level, implying the CSW program should work better at loop level for SYM compared to YM.

In this letter we formulate such a field redefinition using the N=4 SYM light-cone Lagrangian. We proceed in two ways, first we try to formulate a general redefinition by simply requiring the self-dual part of the SYM lagrangian becomes free in the new Lagrangian. Subtleties arise when using it to compute scattering amplitudes that requires one to take into account the contribution of currents under field redefinition. Latter, we will impose the redefinition to be canonical. In both cases only the redefinition of the chiral field is needed, thus giving the transformations for components in a compact manner. However, it is the second redefinition that corresponds to CSW program, and we will see that once stripped away of the superpartners, it gives the result for YM derived in [15]. We calculate the on-shell amplitude in the new lagrangian for 4 pt MHV amplitude and show that it matches the simple form derived in [17]. In the end we briefly discuss the relation between the off-shell MHV vertices here and the on-shell form, with off-shell continuation for propagators , used in CSW.

2 $\mathcal{N} = 4$ light-cone superspace

Without auxiliary fields susy algebra closes only up to field equations. For N=4 SYM for off-shell closure one needs infinite number of auxiliary fields which is still an area of ongoing research. However working with only on-shell degrees of freedom it is possible to manifest half of the susy. Consider N=1 SYM in d=10

$$\mathcal{L} = \frac{1}{g_{10}^2} tr\left(\frac{1}{2}F^a_{MN}F^{aMN} + \bar{\psi}\Gamma^M D_M\psi\right) \quad M,N = 0,1,\cdots 9 \tag{2.1}$$

with transformation rules

$$\delta A_M = \bar{\epsilon}\Gamma_M\psi \ ; \qquad \delta\psi = -\frac{1}{2}F_{MN}\Gamma^{MN}\epsilon \tag{2.2}$$

Consider the two subsequent susy transformation on the spinor

$$\begin{aligned}(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_2}\delta_{\epsilon_1})\psi &= -\frac{1}{2}(2\bar{\epsilon}_2\Gamma_N D_M\psi)\Gamma^{MN}\epsilon_1 - (1 \leftrightarrow 2) \\ &= (\bar{\epsilon}_2\Gamma^P\epsilon_1)D_P\psi - \frac{1}{2}(\bar{\epsilon}_2\Gamma^P\epsilon_1)\Gamma_P\Gamma^M D_M\psi \end{aligned} \tag{2.3}$$

This closes up to the field equation $\Gamma^M D_M\psi = 0$. At this point one can still retain half of the susy on-shell.[1] In a frame where only p^+ is nonvanishing, the Dirac equation is solved if $\Gamma^- p_- \psi = -\Gamma^- p^+ \psi = 0$ where $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^1)$. This means that if one split the spinor ψ into

$$\psi = -\frac{1}{2}(\Gamma^+\Gamma^- + \Gamma^-\Gamma^+)\psi \equiv \psi^+ + \psi^- \tag{2.4}$$

an on-shell spinor means that one has only ψ^-, or $\Gamma^+\psi$ the "+" projected spinor. Looking back at (2.3) indeed the susy algebra with ϵ^+ closes on ψ^-. From the transformation of A_M one sees that only the transverse direction transform under this reduced susy($\delta A_\pm = (\bar{\epsilon}_+\Gamma_\pm\psi_-) = 0$ since $\Gamma^-\psi^- = \Gamma^+\epsilon^+ = 0$). This is the basis for light-cone superfield formalism [16], where half of the susy is manifest with the on-shell degrees of freedom, $A_\perp$ and ψ^-. The susy algebra one is left with is

$$\{Q_{\alpha+}, \bar{Q}^\beta_+\} = (\gamma_+)_\alpha{}^\beta p^+ \tag{2.5}$$

Preserving half of the susy means that only the SO(8) subgroup of the original Lorentz group is manifest. Dimensionally reduce to four dimensions breaks the SO(8) into SO(6) × SO(2) ∼ SU(4) × U(1). The four dimension algebra is then

$$\{\bar{q}^A, q_B\} = -\sqrt{2}\delta^A_B p^+ \tag{2.6}$$

where A, B are SU(4) index, there are 4 complex supercharges. One can then define covariant derivatives with anti-commuting grassman variables, θ^A such that the susy generators

[1] To be more precise, all susy are sill present, although half is manifested linearly and the other half non-linearly. Superspace is only useful for linear representation of susy transformation, which will be our aim here.

and covariant derivatives are given by

$$\bar{q}^A = -\frac{\partial}{\partial\bar{\theta}_A} - \frac{i}{\sqrt{2}}\theta^A\frac{\partial}{\partial x^-}\ ; \qquad \bar{d}^A = -\frac{\partial}{\partial\bar{\theta}_A} + \frac{i}{\sqrt{2}}\theta^A\frac{\partial}{\partial x^-}$$
$$q_A = \frac{\partial}{\partial\theta^A} + \frac{i}{\sqrt{2}}\bar{\theta}_A\frac{\partial}{\partial x^-}\ ; \qquad d_A = \frac{\partial}{\partial\theta^A} - \frac{i}{\sqrt{2}}\bar{\theta}_A\frac{\partial}{\partial x^-}$$

The four dimensional physical fields $\{A, \lambda^A, \phi^{AB}, \bar{\lambda}_A, \bar{A}\}$ transforms as the $\{1, 4, 6, \bar{4}, 1\}$ of SU(4). It is then natural to incorporate them in a scalar superfield, a chiral superfield

$$\bar{d}^A\Phi = 0 \tag{2.7}$$

For N=4 SYM it's multiplet is TCP self-conjugate, therefore there is a further constraint on the chiral fields.

$$\bar{\Phi} = \frac{1}{48(\partial^+)^2}\epsilon^{ABCD}d_Ad_Bd_Cd_D\Phi \tag{2.8}$$

which reflects the self-duality relationship of the scalar fields. Expanding in components

$$\begin{aligned}\Phi(x,\theta) = &\frac{1}{\partial^+}A(y) + \frac{i}{\partial^+}\theta^A\Lambda_A(y) + i\frac{1}{\sqrt{2}}\theta^A\theta^B\bar{C}_{AB}(y)\\ &+\frac{\sqrt{2}}{3!}\theta^A\theta^B\theta^C\epsilon_{ABCD}\bar{\Lambda}^D(y) + \frac{1}{12}\theta^A\theta^B\theta^C\theta^D\epsilon_{ABCD}\partial^+\bar{A}(y)\end{aligned} \tag{2.9}$$

Where $y = (x^+, x^- - \frac{i}{\sqrt{2}}\theta^A\bar{\theta}_A, x, \bar{x})$ and p^+ appears such that each term is dimensionless. The 4 d action can then be written as

$$\begin{aligned}S = tr\int d^4xd^4\theta d^4\bar{\theta}\Bigg\{&\bar{\Phi}\frac{\partial^+\partial^- - \bar{\partial}\tilde{\partial}}{\partial^{+2}}\Phi + \frac{2}{3}gf^{abc}\left[\frac{1}{\partial^+}\bar{\Phi}^a\Phi^b\bar{\partial}\Phi^c + \text{complex conjugate}\right]\\ &-\frac{g^2}{2}f^{abc}f^{ade}\left[\frac{1}{\partial^+}(\Phi^b\partial^+\Phi^c)\frac{1}{\partial^+}(\bar{\Phi}^d\partial^+\bar{\Phi}^e) + \frac{1}{2}\Phi^b\bar{\Phi}^c\Phi^d\bar{\Phi}^e\right]\Bigg\}\end{aligned} \tag{2.10}$$

One can now use (2.8) to transform the action to depend only on the chiral superfield (chiral basis) at the expense of introducing covariant derivatives in the interacting terms. Note however the "self-dual" part of the action can be written in terms of only Φ quite easily.

3 The field redefinition

After transforming (2.10) to the chiral basis, one arrives at a quadratic term, a three pt vertex with 4 covariant derivatives, a three pt and four pt vertex with 8 covariant derivatives. As shown by Chalmers and Seigel [18], the quadratic term and the three point vertex which contains only 4 covariant derivatives describes self-dual SYM. Since self-dual SYM is free classically, at tree level one should be able to consider the the self-dual sector to be simply a free action in the full SYM, i.e. one considers the full SYM as an perturbative expansion around the self-dual sector. Therefore the aim is to redefine the chiral field so that the self-dual sector transforms into a free action: one then tries to find $\Phi(\chi)$ such that

$$\begin{aligned}S_{SD} &= tr\int d^4xd^4\theta\ \left\{\Phi\partial^+\partial^-\Phi - \Phi\tilde{\partial}\bar{\partial}\Phi + \frac{2}{3}\partial^+\Phi[\Phi,\bar{\partial}\Phi]\right\}\\ &= tr\int d^4xd^4\theta\ \{\chi\partial^+\partial^-\chi - \chi\tilde{\partial}\bar{\partial}\chi\}\end{aligned} \tag{3.1}$$

Note that if the field redefinition does not contain covariant derivatives, the remaining interaction terms will becomes MHV vertices, the infinite series generated by the field redefinition from the remaining 3 and 4 pt vertex will all have 8 covariant derivatives. This result is implied by the known MHV amplitude [17]

$$A(\dots j^- \dots\dots i^- \dots)_{\rm tree} = \frac{\delta^8(\sum_{i=1}^n \lambda_i \theta_i^A)}{\Pi_{i=1}^n < i i+1 >}$$

where

$$\delta^8\left(\sum_{i=1}^n \lambda_i \theta_i^A\right) = \frac{1}{2}\prod_{A=1}^4 \left(\sum_{i=1}^n \lambda_i^\alpha \theta_i^A\right)\left(\sum_{i=1}^n \lambda_{i\alpha} \theta_i^A\right) \tag{3.2}$$

The amplitude contains various combination of 8 θs and thus imply 8 covariant derivatives to extract the amplitude.

In the Yang-Mills MHV lagrangian [14, 15], the positive helicity gauge field A transforms into a function of only the new positive helicity field B, while the negative helicity $\bar{A}$ transform linearly with respect to $\bar{B}$, $\bar{A}(\bar{B}, B)$. One can see this result by noting that in order to preserve the equal time commutation relationship,

$$[\partial^+ \bar{A}, A] = [\partial^+ \bar{B}, B] \tag{3.3}$$

that is, the field redefinition is canonical. This implies $\partial^+ \bar{A} = \partial^+ \bar{B}\frac{\delta B}{\delta A}$, therefore $\bar{A}$ transform into one $\bar{B}$ and multiple B fields. This result for the gauge fields becomes natural in the N=4 framework since now the chiral field Φ is redefined in terms of series of new chiral field χ. The positive helicity gauge field A which can be defined in the superfield as $\frac{A}{\partial^+} = \Phi|_{\theta=0} = \Phi(\chi|_{\theta=0})$ resulting in a function that depends only on B. For the negative helicity $\partial^+ \bar{A} = D^4 \Phi|_{\theta=0} = \dots..\chi(D^4\chi)\chi|_{\theta=0}\dots$, dropping contributions from the super partners we see that $\bar{A}(\bar{B}, B)$ depends on $\bar{B}$ linearly.

Another advantage of working with superfields is that as long as the field redefinition does not contain covariant derivatives, the super determinant arising from the field redefinition will always be unity due to cancellation between bosonic and fermionic contributions. Therefore there will be no jacobian factor arising.

The requirement that the field redefinition must be canonical is necessary for the equivalence between MHV lagrangian and the original lagrangian in the framework of the LSZ reduction formula for scattering amplitudes. Indeed we will illustrate this fact by solving the field redefinition for (3.1) disregarding the canonical constraint. We will show that this gives a solution that by itself does not give the correct form of MHV amplitude on-shell, one needs to incorporate the change induce on the external currents. After imposing the canonical constraint we derive the correct on-shell result.

3.1 Field redefinition I $\Phi(\chi)$

We proceed by expanding Φ in terms of χ. Since the light-cone action in the component language corresponds to choosing a light-cone gauge, the redefinition should be performed on the equal light-cone time surface to preserve the gauge condition. We thus Fourier

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transform the remaining three coordinate into momentum space, leaving the time direction alone understanding that all fields are defined on the same time surface.

$$\Phi(\vec{p}_1) = \chi(1) + \sum_{n=2}^{\infty} \int_{\vec{p}_2\vec{p}_3..\vec{p}_{n+1}} C(\vec{p}_2, \vec{p}_3 \cdots \vec{p}_{n+1})\chi(2)\chi(3)..\chi(n+1)\delta\left(\vec{p}_1 + \sum_{i=2}^{n+1} \vec{p}_i\right) \tag{3.4}$$

Here we follow the simplify notation in [15], the light-cone momentums are labelled $p = \{p^-, p^+, \tilde{p}, \bar{p}\}$, the later spatial momentums are collected as a three vector $\vec{p}$ and introduce abbreviation for the momentum carried by the fields, $\chi(i) = \chi(-\vec{p}_i)$. Plugging into (3.1) the coefficient in front of the first term is determined by equating terms quadratic in χ on the left hand side with the right. Similarly for cubic terms we have:

$$\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)tr\int d^4\theta \int_{\vec{p}_2\vec{p}_3\vec{p}_1} \left[-2C(\vec{p}_2, \vec{p}_3)P^2_{2,3} + \frac{2}{3}(p^+_3\bar{p}_2 - p^+_2\bar{p}_3)\right]\chi(1)\chi(2)\chi(3) = 0 \tag{3.5}$$

Thus we have

$$C(\vec{p}_2, \vec{p}_3) = -\frac{1\{23\}}{3P^2_{2,3}} \tag{3.6}$$

Where $P^2_{i..j} = (p_i + \ldots p_j)^2$, $\{i,j\} = p^+_i\bar{p}_j - p^+_j\bar{p}_i$, and for later $(i,j) = p^+_i\tilde{p}_j - p^+_j\tilde{p}_i$.

For four field terms:

$$\begin{aligned}\delta(\Sigma^5_{i=2}\vec{p}_i)tr\int d^4\theta \int_{\vec{p}_2\cdot\cdot\vec{p}_5} \Big[&-C(\vec{p}_2, \vec{p}_3)C(\vec{p}_4, \vec{p}_5)P^2_{2,3} - 2C(\vec{p}_2, \vec{p}_3, \vec{p}_4)P^2_{2,3,4} \\ &-\frac{2}{3}C(\vec{p}_2, \vec{p}_3)\{4,5\} - \frac{2}{3}C(\vec{p}_3, \vec{p}_4)\{(3,4),5\} - \frac{2}{3}C(\vec{p}_4, \vec{p}_5)\{3,(4,5)\}\Big]\chi(2)\chi(3)\chi(4)\chi(5) = 0\end{aligned} \tag{3.7}$$

Using our solution for $C(\vec{p}_2, \vec{p}_3)$ from (3.6), cyclic identity within trace and relabelling the momentums for the last three terms we have:

$$C(\vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{5}{18}\frac{\{2,3\}\{4,5\}}{P^2_{2,3,4}P^2_{2,3}} \tag{3.8}$$

One can again use this result to obtain higher terms iteratively. The field redefinition does not contain covariant derivatives, thus guarantees the remaining vertex after field redefinition will be only of MHV vertex. However if we directly use the new vertices to calculate on-shell amplitude we find that it will differ from the original amplitude computed using the old action. In the next subsection we use YM to illustrate the discrepancy and it's remedy.

3.2 Field redefinition I for YM

One can easily follow the above procedure to solve YM field redefinition.[2] Again we have:

$$tr\int d^4x\ \bar{A}\partial^+\partial^- A - \bar{A}\bar{\partial}\tilde{\partial}A - \frac{\bar{\partial}}{\partial^+}A[A, \partial^+\bar{A}] = tr\int d^4x\ \bar{B}\partial^+\partial^- B - \bar{B}\bar{\partial}\tilde{\partial}B \tag{3.9}$$

We can choose to leave $\bar{A}$ alone, $\bar{A} = \bar{B}$. Following steps similar to the above, for the next to linear term one have:

$$A(1) = B(1) + \int_{\vec{p}_2\vec{p}_3} C(\vec{p}_2, \vec{p}_3)B(2)B(3)\delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\ldots.. \tag{3.10}$$

[2]This redefinition was also investigated in [19].

With

$$C(\vec{p}_2, \vec{p}_3) = \frac{ip_1^+\{2,3\}}{p_2^+ p_3^+ P_{2,3}^2} \tag{3.11}$$

One can then use this result to compute a four point MHV amplitude. With the momentum being on shell now one has

$$C(\vec{p}_2, \vec{p}_3) = \frac{ip_1^+}{(2,3)} \tag{3.12}$$

To see that this does not give the correct result, note that (3.12) is exactly the required redefinition, $\Upsilon(123)$, for A field derived [15]. However, in [15] there is also a field redefinition for $\bar{A}$ while in our approach we left it alone, thus it is obvious that our redefinition will not give the correct on-shell MHV amplitude. The difference between our approach and [15] is the lacking of canonical constraint of the field redefinition. One might guess the discrepancy comes from the jacobian factor in the measure generated by our redefinition (which will be present for YM). However these only contribute at loop level. It is peculiar that field redefinition in the lagrangian formulism should be submitted to constraints in the canonical formulism. Direct comparison for the four pt MHV (- -++) we see that we reproduce the last two terms in eq. (3.13) [15] while the first two terms are missing, the two terms coming from the result of redefining the the $\bar{A}$ field.

The resolution to the missing terms comes from new contribution arising from the currents. In a beautiful discussion of field redefinitions in lagrangian formulism [20], it was pointed out that since scattering amplitudes are really computed in the lagrangian formulism with currents, one should also take into account the effect of the field redefinition for the currents. In the LSZ reduction formula for amplitude, one connects the source to the Feynman diagrams being computed through propagators and then amputate the propagator by multiplying p^2 and taking it on-shell. For YM the currents are $J\bar{A}$ and $\bar{J}A$ where J carries the A external field and $\bar{J}$ carries the $\bar{A}$ field, as can be seen by connecting them to $\langle A\bar{A}\rangle$ propagator. When performing a field redefinition the coupling of the current with the new fields now takes a very different form

$$\bar{J}A(B) \to \bar{J}B + C_2\bar{J}BB + \cdots \tag{3.13}$$

due to these higher order terms, the currents themselves behave as interaction terms. In [15] these higher order contribution vanish after multiplying p^2 and taking it on-shell in the LSZ procedure. In our approach these higher terms will not vanish because of the $\frac{1}{p^2}$ always sitting in front of each field redefinition coefficient as in (3.6), (3.8). Remember the scattering amplitudes are always computed by taking $\frac{\delta}{\delta J}$ (or $\frac{\delta}{\delta \bar{J}}$)of the path integral and multiplying each J (or $\bar{J}$) by p^2 and external wave function, taking everything on-shell in the end. The non-vanishing of the additional terms means we have new contributions to the amplitude.

Adding the contribution of these terms we shall see that one gets the correct amplitude. Consider the 4pt MHV(- -++) or $(\bar{J}\bar{J}JJ)$ amplitude. Now there are four new terms present, two for two different ways of connecting the $\bar{J}BB$ term to the original three pt.vertex, and there is two three point vertex available. A typical graph would be that shown in figure 1,

Consider the three pt vertex $\frac{-i\tilde{p}_2}{p_2^+}p_1^+\bar{B}(k)\bar{B}(2)B(1)$ in the original lagrangian. The $\bar{B}(k)$ leg is now connected to the $\bar{J}BB$ vertex, thus contributing a $\frac{1}{P_{1,2}^2}$. From the LSZ

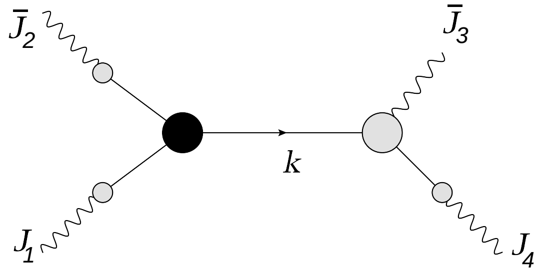

Figure 1. In this figure we show how the field redefinition may contribute to tree graphs from the modification of coupling to the source current. The solid circle indicate the $(--+)$ vertex while the empty circle indicates contraction with the currents. Due to new terms in coupling, the $C\bar{J}_3BB$ term, one can actually construct contribution to the $(--++)$ amplitude by using this term, denoted by the larger empty circle, as a vertex.

procedure there are p^2 multiplying each current. These cancel the remaining propagators except the $\bar{J}$ for the empty circle, the p^2 of that current cancels the $\frac{1}{p^2}$ in front of the field redefinition in (3.11). Putting everything together we have for.

$$-\frac{\tilde{p}_2}{p_2^+}p_1^+\delta(\vec{p}_k+\vec{p}_2+\vec{p}_1)\times\frac{1}{P_{1,2}^2}\times\frac{p_3^+\{-k,4\}}{-p_k^+p_4^+}\delta(\vec{p}_3+\vec{p}_4-\vec{p}_k) \tag{3.14}$$

Using the delta function and putting all external momentum on-shell we arrive at

$$-\frac{\tilde{p}_2p_1^+p_3^{+2}}{p_2^+(p_3^++p_4^+)(3,4)} \tag{3.15}$$

One can proceed the same way to generate other terms by connecting the $\bar{B}(2)$ leg to the $\bar{J}BB$ vertex, and also doing the same thing to the other MHV 3pt vertex $-i\frac{\tilde{p}_kp_2^+}{p_k^+}\bar{B}(k)B(2)\bar{B}(3)$. Collecting everything we reproduce the missing terms. Thus our field redefinition does provide the same on-shell amplitude if we take into the account of contributions coming from the currents.

3.3 Field redefinition II (canonical redefinition)

Due to the extra terms coming from the currents, the field redefinition from the previous sections does not relate to the CSW program, since for CSW the only ingredients are the MHV vertices while above one needs current contribution. In order to avoid complication arising from the currents we impose canonical constraint as in [15], this implies the following relationship

$$tr\int d^4xd^4\theta \quad \Phi(\chi)\partial^+\partial^-\Phi(\chi)=tr\int d^4xd^4\theta \quad \chi\partial^+\partial^-\chi \tag{3.16}$$

This is true because the canonical constraint (3.3) implies that the new field depends on the time coordinate through the old field, there cannot be inverse derivative of time in the coefficients that defines the redefinition. Thus our field redefinition should satisfy (3.16) and

$$tr\int d^4xd^4\theta-\Phi\tilde{\partial}\bar{\partial}\Phi+\frac{2}{3}\partial^+\Phi[\Phi,\bar{\partial}\Phi]=tr\int d^4xd^4\theta\ -\chi\tilde{\partial}\bar{\partial}\chi \tag{3.17}$$

separately. To find a solution to both (3.16) and (3.17) one notes that the component fields are defined in the same way for both chiral superfields, we see that the A field under redefinition will not mixed with other super partners in the supersymmetric theory. Thus we can basically read off the redefinition coefficient from the A field redefinition derived in [15].

$$A(1) = B(1) + \sum_{n=2}^{\infty} -(i)^{n-1} \int_{\vec{p}_2 \cdot\cdot \vec{p}_{n+1}} \frac{p_1^+ p_3^+ .. p_n^+}{(23)(34).(n, n+1)} B(2) \ldots B(n+1) \delta \left(\sum_{i=1}^{n} \vec{p}_i \right) \quad (3.18)$$

The A field redefinition coming from the superfield redefinition in (3.4) would read

$$\frac{A(1)}{ip_1^+} = \frac{B(1)}{ip_1^+} + \sum_{n=2}^{\infty} \int_{\vec{p}_2 \cdot\cdot \vec{p}_{n+1}} C(2, ..n+1)(i)^n \frac{B(2) \ldots B(n+1)}{p_2^+ .. p_{n+1}^+} \delta \left(\sum_{i=1}^{n} \vec{p}_i \right) \quad (3.19)$$

Comparing (3.18) and (3.19) implies the field redefinition for the superfields are

$$\Phi(1) = \chi(1) + \sum_{n=2}^{\infty} \int_{\vec{p}_2 \cdot\cdot \vec{p}_{n+1}} \frac{p_2^+ p_3^{+2} .. p_n^{+2} p_{n+1}^+}{(2,3)(3,4)..(n, n+1)} \chi(2)\chi(3)..\chi(n+1) \delta \left(\sum_{i=1}^{n} \vec{p}_i \right) \quad (3.20)$$

One can check this straight forwardly by computing the redefinition for the $\bar{A}$, stripping away the superpartner contributions gives

$$\bar{A}(1) = \bar{B}(1) + \sum_{n=2}^{\infty} \int_{\vec{p}_2 \cdot\cdot \vec{p}_{n+1}} \sum_{s=2}^{n} (i)^{n+1} \frac{p_s^{+2} p_3^+ p_4^+ .. p_n^+}{p_1^+ (2,3)(3,4)..(n, n+1)} B(2) \ldots \bar{B}(s)..B(n+1) \delta \left(\sum_{i=1}^{n} \vec{p}_i \right) \quad (3.21)$$

this agrees with the result in [15]. It remains to see that the solution in (3.20) satisfy the constraint (3.16) and eq. (3.17). However the fact that the pure YM sector resulting from the super field redefinition satisfies the constraint implies that this is indeed the correct answer. In the appendix we use this solution to prove (3.16) and eq. (3.17) is satisfied. In the next section we use our new field redefinition to reproduce supersymmetric MHV amplitude $\bar{\Lambda}\bar{\Lambda}\Lambda\Lambda$.

3.4 Explicit calculation for MHV amplitude $\bar{\Lambda}\bar{\Lambda}\Lambda\Lambda$

Here we calculate the MHV amplitude in our new lagrangian and compare to know results. For the amplitude $\bar{\Lambda}(1)\bar{\Lambda}(2)\Lambda(3)\Lambda(4)$ we know that the result is

$$\frac{\langle 12 \rangle^2}{\langle 34 \rangle \langle 41 \rangle} \quad (3.22)$$

To transform this into momentum space we follow [15] conventions. For a massless on-shell momentum we write the spinor variables to be:

$$\lambda_\alpha = \begin{pmatrix} \frac{-\tilde{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \qquad \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} \frac{-\bar{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \quad (3.23)$$

Then we have

$$\langle 12 \rangle = \frac{(1,2)}{\sqrt{p_1^+ p_2^+}} \qquad [12] = \frac{\{1,2\}}{\sqrt{p_1^+ p_2^+}} \quad (3.24)$$

Thus (3.22) becomes

$$\frac{(1,2)^2 p_4^+ \sqrt{p_1^+ p_3^+}}{(3,4)(4,1)p_1^+ p_2^+} \tag{3.25}$$

To compute this amplitude from our MHV Lagrangian, we use the relevant field redefinition in components, and then substitute them in the following three and four point vertex of the original Lagrangain.

$$-i\frac{\tilde{\partial}\bar{\mathrm{A}}}{\partial^+}\Lambda\bar{\Lambda} + i\bar{\mathrm{A}}\Lambda\frac{\tilde{\partial}\bar{\Lambda}}{\partial^+} - i\bar{\Lambda}\frac{(\bar{\mathrm{A}}\Lambda)}{\partial^+}\mathrm{A} \tag{3.26}$$

From our field redefinition we can extract the relevant redefinition for $\Lambda\bar{\Lambda}$

$$\begin{aligned}
\Lambda(1) &\to \int_{\vec{p}_2\vec{p}_3} i\frac{(p_2^+ + p_3^+)}{(2,3)}\Lambda'(2)\mathrm{A}'(3)\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3) \\
\bar{\Lambda}(1) &\to \int_{p_2p_3} -i\frac{p_3^+}{(2,3)}\mathrm{A}'(2)\bar{\Lambda}'(3)\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)
\end{aligned} \tag{3.27}$$

Plugging into (3.26) we have five terms. Cyclic rotate the fields to the desired order and relabelling the momentum we arrive at

$$\begin{aligned}
&-\frac{1}{p_2^+ + p_3^+} - \frac{\tilde{p}_2(p_4^+ + p_3^+)}{p_2^+(3,4)} + \frac{\tilde{p}_1(p_4^+ + p_3^+)}{p_1^+(3,4)} - \frac{p_1^+\tilde{p}_2}{(4,1)p_2^+} + \frac{p_1^+(\tilde{p}_2+\tilde{p}_3)}{(4,1)(p_2^+ + p_3^+)} \\
&\quad = -\frac{(1,2)}{(4,1)p_2^+} - \frac{(1,2)(p_4^+ + p_3^+)}{(3,4)p_1^+ p_2^+} = \frac{(1,2)^2 p_4^+}{(3,4)(4,1)p_1^+ p_2^+}
\end{aligned} \tag{3.28}$$

Using the on shell external line factor in light cone for the gauge fields is 1 and for the fermion pair is $\sqrt{p_1^+ p_3^+}$, one reproduces the MHV amplitude in (3.22).

4 CSW-off-shell continuation

An on-shell four momentum can be written in the bispinor form

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} p\bar{p}/p^+ & -\tilde{p} \\ -\bar{p} & p^+ \end{pmatrix} = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}} \quad ; \quad \lambda_\alpha = \begin{pmatrix} \frac{-\tilde{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix}, \ \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} \frac{-\bar{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \tag{4.1}$$

For an off-shell momentum the relationship is modified

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}} + z\eta_\alpha\bar{\eta}_{\dot{\alpha}} \quad ; \quad z = p^- - \frac{\tilde{p}\bar{p}}{p^+}, \ \eta_\alpha = \bar{\eta}_{\dot{\alpha}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.2}$$

imposing $p^2 = 0$ we see that $z = 0$ and we are back at (4.1). The spinors λ and $\bar{\lambda}_{\dot{\alpha}}$ are written in terms of $p^+, \tilde{p}, \bar{p}$, so that it can be directly related to amplitudes computed by the light-cone action which only contains these momentum in the interaction vertices. One can then use these spinors for the off-shell lines by keeping in mind that they relate to the momentum through (4.2). To see this one can compute the three point MHV amplitude by looking directly at the 3 point $--+$ vertex from the light-cone action (even though these

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vanish by kinematic constraint, but it is sufficient to demonstrate the equivalence since the three point MHV vertex is part of the ingredient of CSW). The 3pt vertex for light-cone YM reads $i[\bar{A}, p^+ A]\frac{\tilde{p}}{p^+}\bar{A}$, then the amplitude

$$(1^-2^-3^+) = i\left(\frac{\tilde{p}_1}{p_1^+}p_3^+ - p_3^+\frac{\tilde{p}_2}{p_2^+}\right) = -i\frac{p_3^+}{p_2^+p_1^+}(1,2) = -i\frac{p_3^+}{p_2^+p_1^+}\frac{(1,2)^3}{(2,3)(3,1)} \tag{4.3}$$

where in the last equivalence we used $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$. In our definition for the spinors, we have the identity $\langle 1,2\rangle = \frac{(1,2)}{\sqrt{p_1^+p_2^+}}$ we see that

$$-i\frac{p_3^+}{p_2^+p_1^+}\frac{(1,2)^3}{(2,3)(3,1)} = -i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \tag{4.4}$$

Thus using this relation between the spinors and the momentum, one can relate the "on-shell" form (in terms of $\langle ij\rangle$) to it's off-shell value (in terms of momentum).

Now in the CSW approach the spinor for an off-shell momentum is written as $\lambda_\alpha = p_{\alpha\dot{\alpha}}\bar{X}^{\dot{\alpha}}$, where $\bar{X}^{\dot{\alpha}}$ is the complex conjugate spinor from an arbitrary null external line. Since in the previous analysis, one should take the identification in (4.1) to make the connection between the MHV on-shell form and it's off-shell value, for this to work the CSW offshell continuation must be equivalent to our map, that is

$$\begin{pmatrix} p^- & -\tilde{p} \\ -\bar{p} & p^+ \end{pmatrix}\begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix} = \lambda_\alpha = \begin{pmatrix} \frac{-\tilde{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \tag{4.5}$$

this leads to the requirement that $\bar{X}^{\dot{\alpha}} = \frac{1}{\sqrt{p^+}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For an arbitrary null momentum one can always find a frame such that $k_{\alpha\dot{\alpha}} = k^+\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, this leads to $\bar{X}^{\dot{\alpha}} = \sqrt{k^+}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which differs with the desired result by an overall factor $\frac{1}{\sqrt{k^+p^+}}$. This overall factor cancels in the CSW calculation since the propagator always connect two MHV graphs with one side + side and the other − helicity, the + helicity side has a factor $(\sqrt{k^+p^+})^2$ while the negative helicity side $(\sqrt{k^+p^+})^{-2}$.

To see that one the vertices generated by the redefinition can be written in terms of the holomorphic off-shell spinors (4.1), one needs to prove that these vertices will not depend on $\bar{p}$. This was shown in [14] to be true.

Therefore in the MHV lagrangian, all vertices are MHV vertices and this indicates that one should be able to do perturbative calculation simply by computing Feynman graphs with only MHV vertices. Defining the map between momentum and spinor according to (4.1), one can compute arbitrary off-shell amplitude in light-cone gauge in terms of momentum, and then map to their spinor form. Their spinor form will then take the well known holomorphic form via Nair. The difference between off-shell and on-shell is then incoded in how these spinors relate to their momentum. In a suitable basis, we see that the CSW definition for the spinor is equivalent to our on-shell off-shell map up to an overall factor that cancels in the calcualtion.

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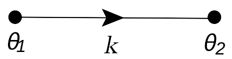
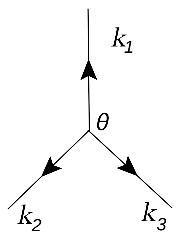
5 Equivalence theorem at one-loop

Again for this to be a proof of the CSW approach, one needs to show that the field redefinition does not introduce new terms that will survive the LSZ procedure and contribute to amplitude calculations. As discussed previously, at tree level all terms generated from the field redefinition of the coupling to source current will cancel through the LSZ procedure except the linear term. The only other possibility will be the self-energy diagram where multiplying by p^2 cancels the propagator that connects this diagram to other parts of the amplitude, and thus surviving. The argument that it vanishes follows closely along the line of [15], one should be able to prove with the requirement of Lorentz invariance that all the loop integrals will be dependent only on the external momentum p^2 which we take to zero in the LSZ procedure. This implies that the self-energy diagrams are scaleless integrals and thus vanish.[3]

We would like to compute the self-energy diagram in light-cone superspace. The Feynman rules for light-cone superspace are defined for the chiral superfield Φ, thus one uses (2.8) to convert all the $\bar{\Phi}$ into Φ. The rules have been derived in [21], and here we simply use the result.[4]



$$\sim \frac{\bar{d}_1^4(k)}{k^2}\delta^8(\theta_1-\theta_2)$$



$$\sim \int d^4\theta d^4\bar{\theta}\frac{d^4(p_1)}{p_1^{+2}}\left[\frac{1}{p_2^+}\frac{\tilde{p}_3 d^4(p_3)}{p_3^{+2}}-\frac{\tilde{p}_2 d^4(p_2)}{p_2^{+2}}\frac{1}{p_3^+}\right] \tag{5.1}$$

Here $d(k)=\frac{\partial}{\partial\theta^A}-\frac{k^+}{\sqrt{2}}\bar{\theta}_A$. The relevant graphs is now shown in figure 2.

Note that other graphs can be manipulated in to the same form by partial integrating the fermionic derivatives. Using (3.20) with $n=2$, the two terms give

$$\int d^4\theta d^4\bar{\theta}J\left[\frac{k^{+2}(\tilde{k}+\tilde{p})}{(k,p)k^2(k+p)^2(k^++p^+)}-\frac{k^+\tilde{k}}{(k,p)k^2(p+k)^2}\right]\Phi=\int d^4\theta d^4\bar{\theta}J\left[\frac{k^+}{k^2(k+p)^2(k^++p^+)}\right]\Phi \tag{5.2}$$

Writing in Lorentz invariant fashion we introduce a light-like reference vector μ in the $+$ direction. The result is rewritten as

$$\int d^4\theta d^4\bar{\theta}J\left[\frac{(k\cdot\mu)}{k^2(k+p)^2(k+p)\cdot\mu}\right]\Phi \tag{5.3}$$

[3]There is of course the question of whether dimensional regularization is the correct scheme for this approach. However since in [22] dimensional regularization was used to give the correct one loop amplitudes from Yang-Mills MHV Lagrangian, the analysis here should hold. However, in [23] a different scheme was used, and it would be interesting to see if there will be any equivalence theorem violation within this scheme.

[4]Note that the propagators given here has already included the factor of $\bar{d}^4$ from the functional derivative $\frac{\delta\Phi(x_1,\theta_1,\bar{\theta}_1)}{\delta\Phi(x_2,\theta_2,\bar{\theta}_2)}=\frac{\bar{d}_1^4}{(4!)^2}\delta^4(x_1-x_2)\delta^4(\theta_1-\theta_2)\delta^4(\bar{\theta}_1-\bar{\theta}_2)$.

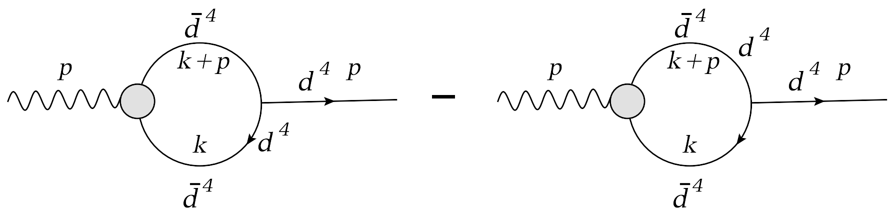


Figure 2. These are the two relevant contribution to the one-loop self-energy diagram. For simplicity we only denote the positions of d^4 and $\bar{d}^4$ to indicate which legs of the vertex was used for the loop contraction.

Again following [15], since by rescaling $\mu \to r\mu$ the factor cancels, thus the resulting integral can only depend on p^2. Since we take $p^2 \to 0$ in LSZ reduction this means that the integral becomes a scaleless integral, and vanishes in dimensional regularization.

6 Discussion

We've shown that by redefining the chiral superfield such that the self-dual part of N=4 SYM becomes free, one generates a new lagrangian with infinite interaction terms which are all MHV vertex. When restricting to equal time field redefinitions the the solution gives the suitable off-shell lagrangian that corresponds to the CSW off-shell continuation. The redefinition is preformed by requiring the self-dual part of the action becomes free since the self-dual sector is essentially free classically. It does not, however, give a derivation of Nairs holomorphic form of n-point super MHV amplitude. For this purpose it is more useful to start from an action that was directly written in twistor space. Indeed such an action has been constructed in [24] and it's relation to CSW has been discussed.

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A Proof of field redefinition

Here we will prove that our field redefinition introduced in section (3.3) satisfies both (3.16) and (3.17). We first produce the proof at leading, terms with three new fields on the l.h.s. of both equations vanish. From this experience we will then show that the same holds for all higher order terms, namely, written in terms of new fields, terms that are more than quadratic in χ on l.h.s. of these equations vanish.

For (3.16) terms with three field comes from the second order term in the field redefinition, namely $\phi(1) \to C(2,3)\chi(2)\chi(3)$ with $C(2,3) = \frac{p_2^+ p_3^+}{(2,3)}$, they give

$$tr \int_{\vec{p}_1\vec{p}_2\vec{p}_3} p^- \left[\frac{p_1^+ p_2^+ p_3^+}{(1,2)} - \frac{p_1^+ p_2^+ p_3^+}{(2,3)}\right] \Phi(1)\Phi(2)\Phi(3)\delta(\Sigma_i \vec{p}_i) \tag{A.1}$$

Using momentum conservation, $(1,2) = -(3,2) = (2,3)$, these two terms indeed cancel each other. The 3 field term that is generated on the l.h.s. for (3.17)

$$tr \int_{\vec{p}_1\vec{p}_2\vec{p}_3} -\frac{p_2^+ p_3^+ p_1 \bar{p}_1}{(2,3)} \chi(1)\chi(2)\chi(3) + \frac{(\bar{p}_2 p_3^+ - \bar{p}_3 p_2^+)}{3} \chi(1)\chi(2)\chi(3) \tag{A.2}$$

Using cyclic identity and relabelling the momentum for the first term we have

$$\begin{aligned}
&tr \int_{\vec{p}_1\vec{p}_2\vec{p}_3} -\chi(1)\chi(2)\chi(3)\frac{1}{3}\left[\frac{p_2^+ p_3^+ \tilde{p}_1 \bar{p}_1}{(2,3)} + \frac{p_1^+ p_2^+ \tilde{p}_3 \bar{p}_3}{(1,2)} + \frac{p_3^+ p_1^+ \tilde{p}_2 \bar{p}_2}{(3,1)}\right] \\
&\quad = tr \int_{\vec{p}_1\vec{p}_2\vec{p}_3} -\chi(1)\chi(2)\chi(3)\left[\frac{p_2^+ p_3^+ \tilde{p}_2 \bar{p}_3 + p_2^+ p_3^+ \tilde{p}_3 \bar{p}_2 - p_2^{+2} \tilde{p}_3 \bar{p}_3 - p_3^{+2} \tilde{p}_2 \bar{p}_2}{3(2,3)}\right] \\
&\quad = tr \int_{\vec{p}_1\vec{p}_2\vec{p}_3} \chi(1)\chi(2)\chi(3)\frac{\{2,3\}}{3}
\end{aligned} \tag{A.3}$$

where in the last two lines we used momentum conservation. This gives the same term as the second term in (A.2) with a minus sign.

To prove that higher field terms also cancel in (3.16) for our field redefinition, note that for n-fields the coefficients combine into

$$\sum_{j=3}^{n-1} C(2, \cdot\cdot, j) p^+_{(j+1,n)} C(j+1, \cdot\cdot, n) = -\frac{(\prod_{i=2}^n p_i^+)(\sum_{j=3}^{n-2} S_j)}{(2,3)(3,4)\cdot\cdot(n,n-1)} \tag{A.4}$$

where we've used the notation that $p^+_{(1,n)} \equiv \sum_{i=1}^n p_i^+$ and

$$S_j \equiv p^+_{n-j} \cdot \cdot p_4^+ p_3^+ [p^+_{n-1} \cdot \cdot p^+_{n+3-j} p^+_{n+2-j} (n+1-j, n-j) + \text{cyclic rotations}] \tag{A.5}$$

For example for $n = 7$

$$\begin{aligned}
S_3 &= p_4^+ p_3^+ \left[p_6^+(5,4) + p_5^+(4,6) + p_4^+(6,5)\right] \\
S_4 &= p_3^+ \left[p_6^+ p_5^+(4,3) + p_5^+ p_4^+(3,6) + p_4^+ p_3^+(6,5) + p_3^+ p_6^+(5,4)\right] \\
S_5 &= \left[p_6^+ p_5^+ p_4^+(3,2) + p_5^+ p_4^+ p_3^+(2,6) + p_4^+ p_3^+ p_2^+(6,5) + p_3^+ p_2^+ p_6^+(5,4) + p_2^+ p_6^+ p_5^+(4,3)\right]
\end{aligned} \tag{A.6}$$

The important point is since these S_j are cyclic sums over terms that are partially anti-symmetric, $S_j = 0$. Hence we've proven that (3.16) is indeed satisfied.

Moving on to (3.17), we use the fact that since (3.16) is satisfied, this implies that[5]

$$\partial^+ \Phi = \frac{\delta \chi}{\delta \Phi} \partial^+ \chi. \tag{A.7}$$

[5] Written in this form we neglect the superspace delta functions and spinor derivatives that usually arises, since we know that the chiral superfield Φ is now already written in terms of chiral superfield χ.

From the discussion above we see that this is indeed true. Plugging back into 3.17 we have

$$\frac{1}{\partial^+}[\partial^+\Phi,\bar{\partial}\Phi] = -\frac{\bar{\partial}\tilde{\partial}}{\partial^+}\Phi + \frac{\delta\Phi}{\delta\chi}\frac{\bar{\partial}\tilde{\partial}}{\partial^+}\chi \tag{A.8}$$

Fourier transform into momentum space and plugging in (3.20) we have

$$\begin{aligned}\left(-\frac{\tilde{p}_1\bar{p}_1}{p_1^+}+\sum_{i=2}^{n}\frac{\tilde{p}_i\bar{p}_i}{p_i^+}\right)C(2,3,\cdots,n) &= \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,\cdots,j)C(j+1,\cdots,n)\{p_{(j+1,n)},p_{(2,j)}\}\\ &= \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)\frac{(j,j+1)}{p_j^+p_{j+1}^+}\{p_{(j+1,n)},p_{(2,j)}\}\end{aligned} \tag{A.9}$$

again $\{p_{(j+1,n)},p_{(2,j)}\} = p^+_{(j+1,n)}\bar{p}_{(2,j)} - \bar{p}_{(j+1,n)}p^+_{(2,j)}$. Since $\frac{(j,j+1)}{p_j^+p_{j+1}^+} = \frac{\tilde{p}_{j+1}}{p^+_{j+1}} - \frac{\tilde{p}_j}{p^+_j}$ the r.h.s. becomes

$$\begin{aligned}&\frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)\left[\frac{\tilde{p}_{j+1}}{p^+_{j+1}}-\frac{\tilde{p}_j}{p^+_j}\right]\{p_{(j+1,n)},p_{(2,j)}\}\\ &\quad= \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)\frac{\tilde{p}_j}{p_j^+}\left[\{p_{(j,n)},p_{(2,j-1)}\}-\{p_{(j+1,n)},p_{(2,j)}\}\right]\\ &\quad= \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)\frac{\tilde{p}_j}{p_j^+}\{1,j\}\end{aligned} \tag{A.10}$$

momentum conversation then gives the l.h.s. of (A.9).

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